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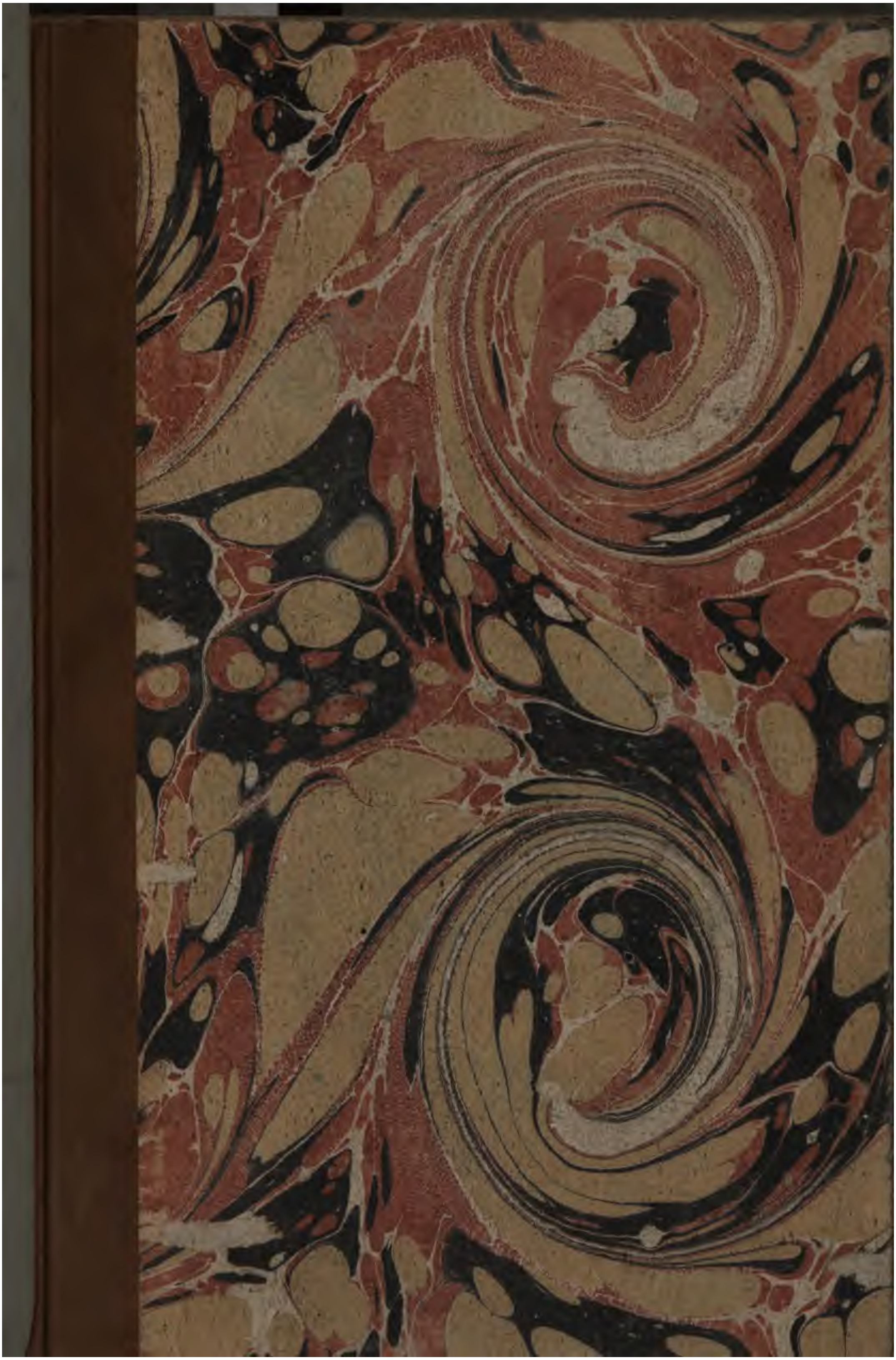

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THE

FIFTH BOOK

OF

EUCLID

ALGEBRAICALLY PROVED

AND

SIMPLIFIED.

DUBLIN
WILLIAM CURRY, JUN. AND COMPANY,
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PREFACE.

THE knowledge of the Fifth Book of Euclid is very necessary for many of the most important investigations in mathematical science.

But the manner in which it is generally proved, is exceedingly tedious and obscure, and thereby too difficult for beginners to understand, on which account it has been excluded from some places of education.

This has induced the writer to try to prove the Propositions in the simplest and briefest manner, which he thinks can be best done by the Algebraical form, as being the method most suited to abstract quantities, and therefore, on such subjects, best adapted to ensure the wished-for brevity and simplicity.

He has considered it desirable, if possible, to solve the Propositions independently of each other, and this he has been enabled to do in almost every instance.

The Algebraical expressions used are extremely simple, and which every learner ought to be acquainted with before commencing Euclid.

THE FIFTH BOOK OF EUCLID.

RATIO is the proportion which one quantity bears to another of the same kind with respect to magnitude.

The magnitude of a ratio is understood by considering how often the first term, called the antecedent, is contained in the other called the consequent. Thus the magnitude of the ratio of $A : B$ is expressed by the fraction $\frac{A}{B}$.

Let the proportion $A : B :: C : D$ express the equality of the ratios $A : B$ and $C : D$; and since the magnitude of the ratio $A : B$ is expressed by the fraction $\frac{A}{B}$, and also that of the ratio of $C : D$ by the fraction $\frac{C}{D}$, it is evident that $\frac{A}{B} = \frac{C}{D}$; or, if four quantities be proportional, the first divided by the second is equal to the third divided by the fourth; and *vice versa* if $\frac{A}{B} = \frac{C}{D}$; $A : B :: C : D$.

Since $\frac{A}{B} = \frac{C}{D}$ it is manifest that $AD = BC$, or if four quantities be proportional, the product of the extremes is equal to the product of the means.

Also if $A : B > C : D$, it is deduced that $\frac{A}{B} > \frac{C}{D}$; and *vice versa* if $\frac{A}{B} > \frac{C}{D}$; $A : B > C : D$.

DEFINITIONS.

I.

A less magnitude is said to be an aliquot part or submultiple of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less; that is, when the greater contains the less a certain number of times exactly.

III.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

IV.

If there be four magnitudes,* and any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever of the second and fourth, if, according as the multiple of the first is greater than the multiple of the second, equal to it, or less, the multiple of the third is also greater than the multiple of the fourth equal to it, or less; then the first is said to have to the second the same ratio that the third has to the fourth.

V.

When of the equimultiples of four magnitudes, taken as in the last definition, the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth, then the first is said to have to the second a greater ratio than the third has to the fourth; and on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

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* Two magnitudes A and B are said to be *equimultiples* of two others C and D when C and D measure A and B respectively the same number of times.

VI.

Analogy or proportion is the similitude of ratios.

VII.

Proportion consists in three terms at least.

VIII.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second. This is called continued proportion.

IX.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

X.

When there is any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on to the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if $A : B :: E : F$; and $B : C :: G : H$; and $C : D :: K : L$; then, by this definition, A is said to have to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if $M : N :: A : D$; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios E to F, G to H, and K to L.

XI.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

Geometers make use of the following technical words, to signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals.

XII.

Permutando, or alternando, by permutation, or alternately; this word is used when there are four proportionals, and it is inferred, that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second is to the fourth. See Prop. XV.

XIII.

Invertendo, by inversion: when there are four proportionals, and it is inferred, that the second is to the first as the fourth is to the third. Prop. B.

XIV.

Componendo, by composition: when there are four proportionals, and it is inferred, that the first, together with the second, is to the second, as the third, together with the fourth, is to the fourth.

XV.

Dividendo, by division: when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. Prop. XVI.

XVI.

Convertendo, by conversion: when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E.

XVII.

Ex æquali (sc. distantia), or ex æquo, from equality of distance: when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: of this there are the two following kinds, which arise from the different order in which the magnitudes are taken two and two.

XVIII.

Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other, and so on in order; and the inference is as mentioned in the preceding definition, whence this is called ordinate proportion Prop. XXI.

XIX.

Ex æquali, in proportione perturbata seu inordinata, from equality, in perturbate or disorderly proportion: this term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last, to the last but two of the second rank, and so on in a cross order; and inference is as in the eighteenth definition. Prop. XXII.

AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

IV.

That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

If any number of magnitudes be equimultiples of as many others, each of each; what multiple soever any one of the first is of its part, the same multiple shall the sum of all the first be of the sum of all the rest.

Let any number of magnitudes A, B, C be equimultiples of as many others, D, E, F each of each; $A + B + C$ is the same multiple of $D + E + F$ that A is of D.

Let A contain D, B contain E, and C contain F, each the same number of times, as for instance, m times. Then because A contains D m times, $A = D + D + D$; D being repeated m times; for the same reason $B = E + E + E +$; E being repeated m times; also $C = F + F + F +$; F being repeated m times. Therefore, adding equals to equals $A + B + C = D + E + F$ taken m times. Therefore, &c. *Q. E. D.*

PROP. II. THEOR.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Suppose the six quantities be A, B, C, D, E, F, and let $A = mB$, $C = mD$, $E = nB$, $F = nD$, then shall $A + E$ be the same multiple of B that $C + F$ is of D.

Since $A = mB$ and $E = nB$, it is evident that $A + E = mnB$; and in like manner, since $C = mD$, and $F = nD$; $C + F = mnD$. Therefore it is manifest that $A + E$ is the same multiple of B that $C + F$ is of D. Therefore, &c. *Q. E. D.*

Prop. III. Theor.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth.

Let A, B, C, D be the four magnitudes, and if $A = mB$ and $C = mD$, it is evident that any equimultiples of A and C will also be equimultiples of mB and mD, or, taking n as an equimultiple of A and C, it is plain that $nA = nmB$, and $nC = nmB$; that is, the equimultiples of the first and second are also equimultiples of the second and fourth. Therefore, &c. *Q. E. D.*

Prop. IV. Theor.

If the first of four magnitudes has to the second the same ratio which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz.—the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.

Let $A : B :: C : D$, and if m and n be any numbers, $mA : nB :: mC : nD$.

Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$, let each side of the equation be multiplied by $\frac{m}{n}$, then shall $\frac{mA}{nB} = \frac{mC}{nD}$. Therefore $mA : nB :: mC : nD$. Therefore, &c. *Q. E. D.*

Prop. V. Theor.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Let $A + B$ be the same multiple of $C + D$ that B is of D; then shall A be the same multiple of C that $A + B$ is of $C + D$.

Let $C + D$ be contained in $A + B$ any number of times, as for instance m times, then $A + B = \overline{C + D} \times m$, or $A + B = mC + m D$, but $B = m D$. Therefore $A = mC$, or A is the same multiple of C that $A + B$ is of $C + D$. Therefore, &c. *Q. E. D.*

Prop. A. Theor.

If the first of four magnitudes have to the second the same ratio which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

Let $A : B :: C : D$, then if $A > B$, $C > D$ if equal, equal; if less, less.

Since $A : B :: C : D$, $AD = BC$. Therefore it is evident that whatever proportion A bears to B, C bears the same to D. Therefore, &c. *Q. E. D.*

PROP. B. THEOR.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

Let $A : B :: C : D$, then shall $B : A :: D : C$.

Since $A : B :: C : D$ $\frac{A}{B} = \frac{C}{D}$, if unity, or one, be now divided by each of the fractions, we have $1 \div \frac{A}{B} = 1 \div \frac{C}{D}$, or $\frac{B}{A} = \frac{D}{C}$. Therefore $B : A :: D$ C. Therefore, &c. *Q. E. D.*

PROP. C. THEOR.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second as the third is to the fourth.

First, let mA and mB be equimultiples of the magnitudes A and B; $mA : A :: mB : B$. It is manifest that $\frac{A}{A} = \frac{B}{B}$, multiply both fractions by m, then $\frac{mA}{A} = \frac{mB}{B}$; wherefore $mA : A :: mB : B$.

Next, let A be the same part of mA that B is of mB; then mA will be the same multiple of A that mB is of B; and therefore, as has been proved, $mA : A :: mB : B$, and inversely $A : mA :: B : mB$. Therefore, &c. *Q. E. D.*

PROP. D. THEOR.

If the first be to the second as the third to the fourth, and if the first be a multiple or part of the second; the third is the same multiple or the same part of the fourth.

Let $A : B :: C : D$; and first let A be a multiple of B, C is the same multiple of D; that is, if $A = mB$, $C = mD$.

Since $A = mB$, substitute mB in the proportion, then $mB : B :: C : D \therefore BC = mBD$ or (taking away B, which is common to both quantities) $C = m\,D$.

Next, let A be a part of B, C is the same part of D; that is, if $mA = B$, $mC = D$, as $mA = B$ let it be substituted in the proportion. Then $A : mA :: C : D \therefore mAC = AD$ or (taking away A, which is common to both quantities) $mC = D$. Therefore, &c. *Q. E. D.*

PROP. VI. THEOR.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Let A and B be equal magnitudes and C any other, $A : C :: B : C$.

Since $A = B$, $\frac{A}{C} = \frac{B}{C} \therefore A : C :: B : C$.

Again, if $A = B$, $C : A :: C : B$; for, as has been proved, $A : C :: B : C$, then inversely $C : A :: C : B$. Therefore, &c. *Q. E. D.*

PROP. VII. THEOR.

Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Let $A + B$ be a magnitude greater than A, and C a third magnitude; $A + B$ has to C a greater ratio than A has to C; and C has to A a greater ratio than it has to $A + B$.

It is manifest that $\frac{A+B}{C} > \frac{A}{C} \therefore A + B : C > A : C$. Again, $C : A > C : A + B$. Now also it is evident that $\frac{C}{A} > \frac{C}{A+B} \therefore C : A > C : A + B$. Therefore, &c. *Q. E. D.*

Prop. VIII. Theor.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

Let $A : C :: B : C$, then $A = B$.

Since $A : C :: B : C$, then $AC = BC \therefore$ dividing both those quantities by C, there remains $A = B$.

Next let $C : A :: C : B$, then $A = B$; as before $AC = BC$, then dividing by C, $A = B$. Therefore, &c. *Q. E. D.*

Prop. IX. Theor.

That magnitude, which has a greater ratio than another has to the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has to another magnitude, is the lesser of the two.

Let $A : C > B : C$, then $A > B$.

Because $A : C > B : C$, $\frac{A}{C} > \frac{B}{C}$ wherefore $A > B$.

Next let $C : B > C : A$ then $B < A$; as $C : B > C : A$, $\frac{C}{B} > \frac{C}{A}$ $\therefore$ $B < A$. Therefore, &c. *Q. E. D.*

Prop. X. Theor.

Ratios that are equal to the same ratio are equal to one another.

If $A : B :: C : D$; and also $C : D :: E : F$; then $A : B :: E : F$.

It is evident that $\frac{A}{B} = \frac{C}{D}$ and $\frac{C}{D} = \frac{E}{F}$ $\therefore$ $\frac{A}{B} = \frac{E}{F}$ or $A : B :: E : F$. Therefore, &c. *Q. E. D.*

Prop. XI. Theor.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so are all the antecedents, taken together, to all the consequents.

Let $A : B :: C : D :: E : F$. Then shall $A : B :: A + C + E : B + D + F$.

It is manifest that $A : B :: A : B :: A : B$. To this ratio add the first ratio, viz.:—$A : B :: C : D :: E : F$, and the result will be $2A : 2B :: A + C : B + D :: A + E : B + F$ or $2A : 2B :: 2A + C + E : 2B + D + F$, then by taking away the common terms there remains $A : B :: A + C + E : B + D + F$. Therefore, &c. *Q. E. D.*

Prop. XII. Theor.

If of six magnitudes, the first have to the second the same ratio which the third has to the fourth, but the third to the fourth, a greater ratio than the fifth has to the sixth; the first has also to the second a greater ratio than the fifth has to the sixth.

If $A : B :: C : D$ but $C : D > E$; F then $A : B > E : F$.

Since $C : D > E : F$, $\frac{C}{D} > \frac{E}{F}$, but also by the first proportion $\frac{A}{B} = \frac{C}{D}$, $\therefore \frac{A}{B} > \frac{E}{F}$ or $A : B > E : F$. Therefore, &c. *Q. E. D.*

PROP. XIII. THEOR.

If the first have to the second the same ratio which the third has to the fourth, and if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

If $A : B :: C : D$ then if $A > C$, $B > D$, if equal, equal, and if less, less.

First let $A > C$; since $\mathrm{A} : B :: C : D$ $\frac{A}{B} = \frac{C}{D}$ then as the fractions are equal, the quotient of $\frac{A}{B}$ is equal to that of $\frac{C}{D}$, let the quotient be m then $\frac{A}{B} = \;$, or $A = mB$, likewise $\frac{C}{D} = m$, or $C = mD$, $\therefore$ as $A > C$; and also $mB = A$, and $mD = C$, mB, $> mD$ or $B > D$.

In the same manner, it may be proved, that if $A = C$, $B = D$; and if $A < C$, $B < D$. Therefore, &c. *Q. E. D.*

PROP. XIV. THEOR.

Magnitudes have the same ratio to one another which their equimultiples have.

If A and B be any two magnitudes, and m any number, $A : B :: mA : mB$.

It is manifest that the fraction $\frac{A}{B} = \frac{A}{B}$, let the numerator and denominator of the fraction on the right side of the equation be multiplied by m, (which will not alter the value of the fraction,) and the result will be $\frac{A}{B} = \frac{mA}{mB}$, $\therefore A : B :: mA : mB$. Therefore, &c. *Q. E. D.*

PROP. XV. THEOR.

If four magnitudes of the same kind are proportionals, they are proportionals also when taken alternately.

If $A : B :: C : D$, then shall $A : C :: B : D$, Since $A : B :: C : D$, $AD = BC$, let both sides of the equation be divided by CD, then $\frac{AD}{CD} = \frac{BC}{CD}$; or, by taking away the terms common to both numerator and denominator of each of the fractions $\frac{A}{C} = \frac{B}{D}$, $\therefore A : C :: B : D$. Therefore, &c. *Q. E. D.*

PROP. XVI. THEOR.

If the first, together with the second, have to the second the same ratio which the third, together with the fourth, has to the fourth, the first will have to the second the same ratio which the third has to the fourth.

If $A+B : B :: C+D : D$, then shall $A : B :: C : D$.

Since $A + B : B :: C + D : D$, $\frac{A+B}{B} = \frac{C+D}{D}$, let unity be subtracted from both sides of the equation, $\frac{A+B}{B} - 1 =^{*} \frac{C+D}{D} - 1$ or $\frac{A}{B} = \frac{C}{D}$, $\therefore A : B :: C : D$. Therefore, &c. *Q. E. D.*

PROP. XVII. THEOR.

If the first be to the second as the third to the fourth, the first plus or minus the second is to the second, as the third plus or minus the fourth is to the fourth.

If $A : B :: C : D$, then shall $A + B : B :: C + D : D$.

Because $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$. Let unity be added to both sides of the equation, then $\frac{A}{B} + 1 = \frac{C}{D} + 1$ or $\frac{A+B}{B} = \frac{C+D}{D}$, $\therefore A+B : B :: C+D : D$.

In the same manner, by subtracting unity from

---

* $\frac{A+B}{B} - 1 = \frac{A+B}{B} - \frac{1}{1}$ let those fractions be brought to a common denominator, viz. $\frac{A+B}{B} - \frac{B}{B}$, then by subtraction $\frac{A}{B}$ will be the result.

$\frac{A}{B}$ and $\frac{C}{D}$, it may be proved that $A - B : B :: C - D : D$. Therefore, &c. *Q. E. D.*

PROP. E. THEOR.

If the first be to the second as the third is to the fourth, the first; the first is to its excess above the second, as the third is to its excess above the fourth.

If $A : B :: C : D$, then $A : A - B :: C : C - D$.

Since $A : B :: C : D$ inversely $B : A :: C : D \therefore \frac{B}{A} = \frac{C}{D}$. If unity be now taken from both sides of the equation, $\frac{B}{A} - 1 = \frac{D}{C} - 1$ or $\frac{A - B}{A} = \frac{C - D}{C} \therefore A - B : A :: C - D : C$ and inversely $A : A - B :: C : C - D$. Therefore, &c. *Q. E. D.*

PROP. XVIII. THEOR.

If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder will be to the remainder as the whole is to the whole.

If $A : B :: C : D$, and if C and D be less than A and B; $A - C : B - D :: A : B$.

As $A : B :: C : D$, alternately $A : C :: B : D$, wherefore $\frac{A}{C} = \frac{B}{D}$, then if unity be taken from both sides of the equation, $\frac{A}{C} - 1 = \frac{B}{D} -$

1, or $\frac{A-C}{C}=\frac{B-D}{D}$ $\therefore A - C : C :: B - D : D$. Again alternately $A - C : B - D :: C : D$, but $C : D :: A : B$. Therefore, $A - C : B - D :: A : B$. Therefore, &c. *Q. E. D.*

Prop. XIX. Theor.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio; if the first be greater than the third, the fourth is greater than the sixth; if equal, equal; and if less, less.

Let there be two series of three magnitudes, viz. A, B, C and D, E, F; now if $A : B :: D : E$, and also $B : C :: E : F$, then if $A > C$, $D > F$; if $A = C$, $D = F$; and if $A \angle C$, $D \angle F$. First let $A > C$, since $A : B :: D : E$, and $B : C :: E : F$ alternately $A : D :: B : E$ and $B : E :: C : F$, then (10) $A : D :: C : F$ $\therefore$ (13), since $A > C$, $D > F$.

In the same manner it may be proved (13) that if $A = C$, $D = F$, and if $A \angle C$, $D \angle F$. Therefore, &c. *Q. E. D.*

Prop. XX. Theor.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth is greater than the sixth; if equal, equal; and if less, less.

Let there be two series of magnitudes, viz. A, B, C and D, E, F, such that $A : B :: E : F$ and $B : C :: D : E$; if $A > C$, $D > F$; if $A = C$, $D = F$, and if $A \angle C$, $D \angle F$.

As $A : B :: E : F \frac{A}{B} = \frac{E}{F}$, also since $B : C :: D : E$, $\frac{B}{C} = \frac{D}{E}$, let $\frac{A}{B}$ be multiplied by $\frac{B}{C}$, as also $\frac{E}{F}$ by $\frac{D}{E}$, then $\frac{A}{B} \times \frac{B}{C} = \frac{E}{F} \times \frac{D}{E}$, or $\frac{AB}{BC} = \frac{ED}{FE}$, or $\frac{A}{C} = \frac{D}{F}$, wherefore $A : C :: D : F$. Therefore (A) if $A \mathrel{\rotatebox[origin=c]{180}{$\angle$}} C$, $D \mathrel{\rotatebox[origin=c]{180}{$\angle$}} F$, if $A = C$, $D = F$, and if $A \angle C$, $D \angle F$. Therefore, &c. *Q. E. D.*

PROP. XXI. THEOR.

If there be any number of magnitudes, and as many others, which taken two and two in order have the same ratio; the first will have to the last of the first magnitudes the same ratio which the first of the others has to the last.

If there be two series of magnitudes, viz. A, B, C and D E, F, which, taken two and two in order, have the same ratio, viz. $A : B :: D : E$, and $B : C :: E : F$, then $A : C :: D : F$.

Because $A : B :: D : E$, $\frac{A}{B} = \frac{D}{E}$, and also since $B : C :: E : F$, $\frac{B}{C} = \frac{E}{F}$, let $\frac{A}{B}$ be multiplied by $\frac{B}{C}$, and also $\frac{D}{E}$ by $\frac{E}{F}$, then $\frac{A}{B} \times \frac{B}{C} = \frac{D}{E} \times \frac{E}{F}$, or $\frac{AB}{BC} = \frac{DE}{EF}$, or $\frac{A}{C} = \frac{D}{F} \therefore A : C :: D : F$. Therefore, &c. *Q. E. D.*

PROP. XXII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio: the first will have to the last of the first magnitudes the same ratio which the first of the others has to the last.

If there be any number of magnitudes, A, B, C, and D, E, F, which taken two and two in a cross order, have the same ratio, viz. $A : B :: E : F$, and $B : C :: D : E$, then shall $A : C :: D : F$. It is manifest that $\frac{A}{B} = \frac{E}{F}$, and also that $\frac{B}{C} = \frac{D}{E}$, if then $\frac{A}{B}$ be multiplied by $\frac{B}{C}$, and $\frac{E}{F}$ by $\frac{D}{E}$; $\frac{A}{B} \times \frac{B}{C} = \frac{E}{F} \times \frac{D}{E}$, or $\frac{AB}{BC} = \frac{ED}{FE}$ or $\frac{A}{C} = \frac{D}{F}$ $\therefore$ $A : C :: D : F$. Therefore, &c. *Q. E. D.*

PROP. XXIII. THEOR.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth, together, shall have to the second, the same ratio which the third and sixth together, have to the fourth.

Let $A : B :: C : D$, and also $E : B :: F : D$, then $A + E : B :: C + F : D$.

It is evident from the two proportions, that $\frac{A}{B} = \frac{C}{D}$, and $\frac{E}{B} = \frac{F}{D}$, let $\frac{A}{B}$ and $\frac{E}{B}$ be added together, as also $\frac{C}{D}$ and $\frac{F}{D}$, then $\frac{A}{B} + \frac{E}{B} = \frac{C}{D} + \frac{F}{D}$, or $\frac{A + E}{B} = \frac{C + F}{D}$ $\therefore A + E : B :: C + F : D$. Therefore, &c. *Q. E. D.*